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Upper Bound for Linear Arboricity

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Abstract. Using the König-Hall Theorem, we establish the Akiyama-Exoo-Harary Conjecture up to an additive factor which is at most linear in the square root of the graph's topological genus.

Terminology and notation are standard; see, e.g., [3] or [7]. Graphs G have no parallel edges or loops; write ΔG for the maximum degree. As usual, $\lceil x \rceil$ is the least integer $\geq x$, while $\lfloor x \rfloor = -\lceil -x \rceil$. For n an integer, $n \bmod 2$ is 0 or 1 according to whether n is even or odd. A function h is *monic* if $hx = hy$ implies $x = y$. If S is a finite set, card(S) is the number of its elements. A forest is *linear* if each of its components is a path, and the *linear arboricity* of a graph G, denoted la(G), is the least number of linear forests in an edge covering of G. One can plainly assume that the linear forests are edge-disjoint. Thus, la$(G) \geq \lceil \Delta G/2 \rceil$. When G is a cycle, the lower bound is not sharp, but Akiyama, Exoo and Harary [1] proposed:

CONJECTURE 1. *For any G, la$(G) \leq \lceil (1 + \Delta G)/2 \rceil$.*

A substantial amount of work has been done on this problem (see [2]). Guldan [5] has proved that for any graph G,

$$\mathrm{la}(G) \leq c + \lceil (6/5) \lfloor \Delta G/2 \rfloor \rceil,$$

where $c = \Delta G \bmod 2$, while Alon [2] has shown that, for any $\epsilon > 0$, there is an integer $N(\epsilon)$ such that whenever $\Delta G \geq N(\epsilon)$, la$(G) \leq (\epsilon + \frac{1}{2})\Delta G$. We settle the conjecture up to an additive constant; see Theorem 3.

Our chief tool is the well-known König-Hall Theorem ([6], [9]). Let $T(1), \dots, T(n)$ be any sequence of sets. A set T is called a *system of distinct representatives*, SDR, if $T = \{t(1), \dots, t(n)\}$ where $t(i) \in T(i)$ and $t(i) \neq t(j)$ for all i, j with $1 \leq i \neq j \leq n$.

THEOREM 1. *Let $T(1), \dots, T(n)$ be any sequence of sets. There is an SDR, if and only if, for $1 \leq u \leq n$, the union of any u members of the sequence has $\geq u$ elements.*

THEOREM 2. *Let $T(1), \dots, T(n)$ be a sequence of sets, each of cardinality at least n. Then there is an SDR.*

Theorem 2 follows immediately from Theorem 1. We also use an interesting graph invariant, *degeneracy*, discovered independently by Matula [11], by Lick and White [10] and also by Szekeres and Wilf [13]. Following our notation in [12], and references cited there, we write sw(G) for the degeneracy of G which is the maximum over all subgraphs H of G of the minimum degree in H.

In [8] we proved a special case of Conjecture 1 when sw$(G) \leq 2$; this includes the class of outerplanar graphs. Here we give a stronger result.

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THEOREM 3. *For any* G, $la(G) \le \lceil (\Delta G + sw(G) - 1)/2 \rceil$.

We showed in [12; pp. 44-45] that for any graph G embeddable in the orientable surface with genus $\gamma > 0$

$$\mathrm{sw}(G) \le -1 + \lfloor (7 + \sqrt{(1 + 48\gamma)})/2 \rfloor.$$

It is elementary to check that $\mathrm{sw}(G) \le 5$ for G planar. Note also that Theorem 3 implies the fact [1] that $\mathrm{la}(G) = \lceil \Delta G/2 \rceil$ when G is a forest. Of course, our result is not useful for the case of d-regular graphs (cf. [2]).

PROOF OF THEOREM 3: Put $m = \mathrm{sw}(G)$ and for any subgraph H of G, let $f(H) = \lceil (\Delta H + m - 1)/2 \rceil$. We shall prove that for any subgraph H of G

$$\mathrm{la}(H) \le f(H). \tag{1}$$

which includes the theorem when $H = G$.

Suppose (1) is false and let H be a minimum-vertex counterexample. Choose v in H so that $\deg(H, v) = r \le m$. Consider $K = H - v$. By hypothesis,

$$K = L(1) + \ldots + L(p), \tag{2}$$

where "+" denotes edge-disjoint union, $L(i)$ is a linear forest, $1 \le i \le p$, and $p \le f(K)$. Allowing trivial forests, we shall assume that $p = f(K)$. Let $v(j)$, $1 \le j \le r$, be the neighbors of v in H, and define

$$d(j) = 2p - \deg(K, v(j)), \quad 1 \le j \le r. \tag{3}$$

For $1 \le j \le r$, $\deg(K, v(j)) = \deg(L(1), v(j)) + \ldots + \deg(L(p), v(j)) \le 2p$; so $d(j) \ge 0$. Let $E(j) = \{i : \deg(L(i), v(j)) = 1\}$ and similarly, let $C(j) = \{i : \deg(L(i), v(j)) = 0\}$, where both are subsets of $\{1, \ldots, p\}$. Clearly, $d(j) = \mathrm{card}(E(j)) + 2 \cdot \mathrm{card}(C(j))$ for $1 \le j \le r$.

Let $S(j)$ be the set defined as the disjoint union of $E(j)$ with two disjoint copies of $C(j)$, say $C(j) \times \{0, 1\}$. Thus, $S(j) = E(j) + C(j) \times \{0, 1\}$ and so $\mathrm{card}(S(j)) = d(j)$, for $1 \le j \le r$.

We shall complete the proof of the theorem by extending the partition (2) of the edges of K into linear forests into a partition of the edges of H, possibly allowing one more linear forest, in such a way that H must satisfy (1), which is impossible.

First, we need a lower bound on the cardinality of the sets $S(j)$. In particular, this lemma tells us that there are always linear forests at $v(j)$ to which the edge $vv(j)$ could be added.

LEMMA 1. *For* $1 \le j \le r$, $d(j) \ge m - c$, *where* $c = (\Delta K + m) \mod 2$.

PROOF: By (3), $d(j) + \deg(K, v(j)) = 2p$ for $1 \le j \le r$. But $\Delta K \ge \deg(K, v(j))$ and $2p = 2f(K) = 2\lceil (\Delta K + m - 1)/2 \rceil = \Delta K + m - c$, where $c = (\Delta K + m) \mod 2$. This suffices. ∎

We divide the remainder of the proof of Theorem 3 into two cases:

[$\Delta K + m$ **even**].

By Theorem 2 and Lemma 1, the collection $S(j)$, $1 \le j \le r$, has an SDR $S = \{s(j) | 1 \le j \le r\}$ since $r \le m$. Recall that the elements of each $S(j)$ consist of linear forests from the decomposition (2) of K, which are of two types: $E(j)$, those ending at $v(j)$ (i.e., which have a path ending at $v(j)$) and $C(j)$, those missing $v(j)$, the latter coming in two "colors". Let $h(S, j)$, or when the SDR S is understood, $h(j)$, be the element in $\{1, \ldots, p\}$ corresponding to $s(j)$. Note that we can have $h(j) = h(k)$ for $j \ne k$ only when the linear forest either misses both $v(j)$ and $v(k)$ or misses one and ends at the other.

Whenever $h(j) = h(k)$ for $j \ne k$, take $i = h(j)$ and put both $vv(j)$ and $vv(k)$ into $L(i)$. Delete i or $\{(i, 0), (i, 1)\}$ from each member of $S(u)$, $1 \le u \le r$, and consider the revised family for $u \ne j$ and $u \ne k$. Again, by Theorem 2, there is an SDR, and the process can be repeated until the function h is monic or all edges $vv(u)$ have been used. At each step, two of the edges at v are added to one of the $L(i)$ so that the result is a linear forest in H. This process is non-unique and depends on the sequence $v(1), \ldots v(r)$. When h is monic, the SDR assigns each remaining edge at v to a distinct linear forest in K, so that the result is a linear forest in H.

Hence, $\mathrm{la}(H) \leq p = f(K) \leq f(H)$; reductio ad absurdam since H was supposed to be a counterexample to (1).

[$\Delta K + m$ odd].

By Lemma 1, either $d(j) \geq m$ for $1 \leq j \leq r$, and one argues exactly as before, or else some $d(j) = m - 1$. Then $\Delta H = 1 + \Delta K$ and $f(H) = 1 + f(K)$. Hence, we need only show $\mathrm{la}(H) \leq 1 + p$. But this is clear. Put the edge $vv(r)$ at v in a separate linear forest, apply Lemma 1 to the family $S(j)$, $1 \leq j \leq r - 1$, and proceed as above. ∎

There is an interesting practical application for our technique. Suppose given a family $J = \{j(1), \ldots, j(p)\}$ of p "tasks" (or "jobs") and a collection $M(1), \ldots, M(r)$ of r "machines". Further suppose that for $1 \leq j \leq r$, $M(j)$ has some tasks $E(j) \subseteq J$, which it *must* control, and others $C(j) \subseteq J$, which it can perform with or without control. Call $C(j) + E(j)$ the set of *feasible jobs* for $M(j)$.

If each machine is assigned a feasible job, if no job is assigned to more than two machines and, if when two machines share a job they do not both require control, then we call the resulting scheme a *shared job assignment* (SJA) with respect to $M(j)$, $C(j)$, $E(j)$, $1 \leq j \leq r$. Our proof of Theorem 3 actually shows the following:

THEOREM 4. *There is an SJA if* $\mathrm{card}(E(j)) + 2\mathrm{card}(C(j)) \geq r$ *for* $1 \leq j \leq r$.

This result generalizes Theorem 2. It is also possible to extend Theorem 1 (and, a fortiori, Theorem 2). For example, Ford and Fulkerson [4] proved a general theorem on "restricted representations" which implies that the equivalent condition for up to r-fold sharing of representatives is $\mathrm{card}(\cup\{T(j) | j \in J\}) \geq \lceil \mathrm{card}(J)/r \rceil$, $1 \leq \mathrm{card}(J) \leq n$. In contrast, our definition of sharing allows each machine to separate the tasks independently, just as $L(i)$ may have paths ending at some $v(j)$ but not at others.

The analogy of machines and tasks corresponding to nodes and linear forests may be useful in modeling parallel computation. The separate paths in each linear forest are a spatial decomposition, while the sequence of forests corresponds to time.

REFERENCES

1. J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs III, *Math. Slovaca* **30**, 405–417 (1980).
2. N. Alon, The linear arboricity of graphs, *Israel J. of Math.* **62**, 311–325 (1988).
3. G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Wadsworth & Brooks, Monterey, CA, (1986).
4. L.R. Ford, Jr. and D.R. Fulkerson, Network flows and systems of representatives, *Canadian J. Math.* **10**, 78–84 (1958).
5. F. Guldan, Some results on linear arboricity, *J. of Graph Theory* **10**, 505–509 (1986).
6. P. Hall, On representatives of subsets, *J. London Math. Soc.* **10**, 26–30 (1935).
7. F. Harary, *Graph Theory*, Addison-Wesley, Reading, (1969).
8. P.C. Kainen, Arboricity and edge partitions, In *Proc. Second Int'l. Conf. in Graph Theory, Combinatorics, Algorithms and Appl.*, Y. Alavi *et al.*, Ed., SIAM (to appear).
9. D. König, *Theorie Der Endlichen Und Unendlichen Graphen*, Reprint of 1935 edition, Chelsea Publ., New York.
10. D.R. Lick and A.T. White, k-degenerate graphs, *Canadian J. Math* **22**, 1082-1096 (1970).
11. D.W. Matula, A min-max theorem for graphs with applications to graph coloring, *SIAM Review* **10**, 481–482 (1968).
12. T.L. Saaty and P.C. Kainen, *The Four-Color Problem*, Dover Press, New York, (1986).
13. G. Szekeres and H.S. Wilf, An inequality for the chromatic number, *J. Combinatorial Theory* **4**, 1–3 (1968).

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